ABSTRACT
The aim of this paper is to make a comparison between an original numerical code based on the finite elements method and a series of experimental data related to the heat transfer coefficient of a sinusoidal profiled dissipator. It is clear that even though the experimental data are interesting and provide a useful indication of values within the limits of their own or of similar application ranges, these must be generalized: the new code performs this function – having validated it also by other comparisons – thus allowing a sufficiently precise estimate of heat transfer coefficients for generation in many different practical cases.

The created code has good versatility (study of geometries, materials, different thermal flows) and can be easily included on those personal computers based on the most widely-used systems at a world level.

1. INTRODUCTION
In [1], which can also be consulted for further possible references, a specially constructed apparatus is described for the experimental determination of the heat transfer coefficient in a dissipator with sinusoidal shaped surface. The reported data are interesting, but it is obvious that their generalisation is essential for a vast use in the complexity of the possible applications – some of which are already mentioned in the same paper [1] – in the field of heat exchangers, electronic component production, nuclear plant engineering, etc. In fact, for industrial apparatus construction, verification must be made during the planning phase, by means of calculation, of the situations that arise.

In the mechanical field of high thermal flows and in the nuclear field, particularly delicate and dangerous conditions may occur, causing the need to establish with precision, prior to or in the planning phase, for example, the heat to be removed in critical zones and/or subject to possible burn-outs, or even enable the containment of the values of the warm channel factors – using coefficients of known local average thermal exchange.

In the present paper, in order to satisfy this technical necessity, an original general code was developed which is first compared to the experimental data introduced in [1, 2, 3, 4] in order to check its validity, after which it may then be applied to other situations with different geometries and with varying thermal flows, while continuing the comparison wherever possible, thus providing the coefficient of convection exchange to be used either prior to planning or during the construction of the apparatus.

2. DESCRIPTION OF THE PRINCIPAL CHARACTERISTICS OF THE CODE OF CALCULATION
The flow diagram of the code is characterized, in addition to the usual blocks, by the ability to consider different thermal flows and medium conductivity possibly variable with the temperature, geometries and variable boundary conditions.

The created code is based on the method of the finite elements (MEF) and resolves the differential equations with functions that describe the temperature, no longer on the entire considered domain, but on sub-domains (finite elements), obtained by transforming the domain of departure into discreet domain.
2.1 The method of the finite elements (FEM)
As is known, the method of the finite elements is a general method for the approximate solution of one or more differential equations to partial derivatives which, in this case, are the equations that govern the steady thermal problem.

2.1.1 The steady thermal problem
The steady thermal problem is considered in a homogeneous and isotropic solid.

The heat transfer equation is the Fourier equation:

\[ \lambda \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + Q(x, y) = 0 \quad (1) \]

in the plane x, y, T being the temperature, \( \lambda \) the conductivity coefficient, Q the heat generation.

The boundary conditions are:

- \( T = T_0 \) (1.1)

- \( -\lambda \frac{\partial T}{\partial n} = H(T - T_\infty) \) (1.2)

- where \( n \) represents the outward-drawn normal to the surface, while H is the heat transfer coefficient and \( T_\infty \) the surroundings temperature.

We also have:

\[ -\lambda \frac{\partial T}{\partial n} = q_0 \quad (1.3) \]

where \( q_0 \) is the imposed flow.

The analogous equation to (1), in the axisymmetric case with co-ordinates \( r, z \), is:

\[ \lambda \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \lambda \frac{\partial^2 T}{\partial z^2} + Qr = 0 \quad (2) \]

with the analogous boundary conditions.

By discretizing the domain of calculation with the MEF, the equation of thermal equilibrium of the single finite element is obtained:

\[ K^e T^e = F^e \quad (3) \]

where:

\[ K^e = K^e_1 + K^e_2 \]

\[ K^e_1 = \int_{A^e} (K B^T B)^e dA \]

\[ K^e_2 = \int_{S^e} (H N^T N)^e dS \]

\[ F^e = F^e_0 + F^e_H + F^e_0 \]
\[ F^e_0 = \int_{A^e} (Q N)^e dA \]

\[ F^e_H = -\int_{S^e} (q N)^e dS \]

and \( N^e \) are the shape functions, \( F^e_0 \) is the vector of the thermal concentrated load (punctiform heat sources), \( B^e \) is the matrix of the Cartesian derivatives of the shape functions.

We also have:

\[ q = q_0 - T H_0 \]

### 2.1.2 Calculation of the thermal load vector

The thermal load relating to the \( F^e \) element is calculated in the subroutine that deals with the calculation of the thermal loads.

The load is defined in a special matrix with indexes I and J, where I indicates the element and J the number of the nodal component.

Four types of load are considered:
- concentrated;
- of generation;
- of heat flow;
- convective.

In the case of the concentrated thermal load \( F^e_0 \), first the number of the loaded nodes are read and then the node with the relative load.

In the case of generation of heat, the value of \( Q \) is taken from a matrix in which the properties of the materials are contained (conductivity coefficient, thickness of the material and heat generation).

The thermal load \( F^e_Q \) is calculated for all the elements by two separate subroutines, one for the triangular elements and another one for the quadrangular elements.

In the case of thermal flow, the \( q_0 \) value is read by the special line of the matrix that contains the properties of the materials.

The thermal load \( F^e_H \) is calculated only for those elements of the boundary that are invested by the thermal flow.

First the number of the loaded walls is read and then the element loaded with the relative nodes.

The calculation of the relative thermal load is carried out in the special subroutine to which the \( FLUX = -q_0 \) value is passed.

In the case of convection, the value of \( H \) is always contained in the matrix of the properties of the materials, as is the value of \( T_\infty \).

The thermal load \( F^e_H \) is calculated only for the elements of the boundary that are affected by the convection and the procedure is analogous to that previously seen for the thermal flow.

The calculation of the relative thermal load is carried out in the special subroutine to which the \( FLUX = H \cdot T_\infty \) value is passed.

The calculation of the integral \( \int_{A^e} (Q N)^e dA \) by numerical means is carried out by the opportune subroutines for the triangular elements and for the quadrangular elements.

In the case of the plane triangular element, the integral becomes, for the generic node \( n \):

\[
F_n = \int_{A^e} s Q N_n \, dx \, dy = \int_{0^1}^{1^1} s Q N_n \, J \, d\xi \, d\eta \sum_{i=1}^{NGP} s Q N_n (\xi_i, \eta_i) J(\xi_i, \eta_i) PES (i)
\]

(4)

and for the plane quadrangular element:

\[
F_n = \int_{A^e} s Q N_n \, dx \, dy = \int_{-1}^{1} \int_{-1}^{1} s Q N_n \, J \, d\xi \, d\eta \sum_{i=1}^{NGP} \sum_{j=1}^{NGP} s Q N_n (\xi_i, \eta_i) J(\xi_i, \eta_i) PES (i) PES (j)(5)
\]

Where \( NGP \) indicates the number of Gauss points.
In the assialsymmetric case, the thickness must be replaced by the value \(2 \pi r (\xi, \eta)\). The value of NGP is the degree of the polynomial to be integrated and it depends on the number of nodes of the element and on the type of problem. Opportune subroutines draw the shape functions, their derivatives and the Jacobian. It should be remembered that in the case of assialsymmetric problems, the radial co-ordinate must be calculated:

\[
r = \sum_{i=1}^{n} N(\xi, \eta_i) r_i
\]

in the Gauss points \(n\) being the number of nodes per element. At the end of the cycle, the generic load is added to the matrix of the loads relating to the element. The calculation of the integral \(\mathbf{F}_e^* = -\int (\mathbf{qN})^T d\mathbf{S}\) by numerical means is carried out by a special subroutine, but since the integral is mono-dimensional in this case, the integration involves the sole variable \(\xi\).

For the generic node \(n\) of the boundary, we have, in the plane case:

\[
\mathbf{F}_n = -\int_S \mathbf{q} \mathbf{N}_n \, d\mathbf{S}
\]  

(7)

The length of a \(d\mathbf{S}\) element of the boundary is \((x, y)\) in the plane case and \((r, z)\) assialsymmetric:

\[
d\mathbf{S} = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} \, d\xi = L \, d\xi
\]

(8)

and therefore, from (8) the nodal load is obtained:

\[
\mathbf{F}_n \equiv \sum_{i=1}^{NGP} s \, q \, N_\gamma(\xi_i) \, L(\xi_i) \, PES(i)
\]  

(9)

In the remaining internal nodes, the corresponding thermal load will be void. With the obvious changes, assialsymmetric cases may also be treated. The calculation of (9) follows the same scheme previously seen. In the case of parabolic assialsymmetric elements with unstraight sides, the integral calculus cannot be but approximate, since the integrating function is not a polynomial.

### 2.1.3 Calculation of the conductivity matrix

The calculation of the integral \(\mathbf{K}_i^* = \int_{A'} (\mathbf{KB}^T \mathbf{B}) d\mathbf{A}\) by numerical means is carried out here too by two distinct subroutines, one for the triangular elements and another one for the quadrangular elements. For a plane triangular element, the integral becomes:

\[
\mathbf{K} = \int_{A} s \, K \, B^T \, B \, d\mathbf{A} = \int_{0}^{1} \int_{0}^{1-\gamma} s \, K \, B^T \, B \, J \, d\xi \, d\eta
\]

(10)

where

\[
\mathbf{B} = \frac{1}{J} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \end{bmatrix}
\]

(11)

set
\[ B_{1i} = +J_{22} \frac{\partial N_i}{\partial \xi} - J_{12} \frac{\partial N_i}{\partial \eta} \]
\[ B_{2i} = -J_{21} \frac{\partial N_i}{\partial \xi} + J_{11} \frac{\partial N_i}{\partial \eta} \]  \hspace{1cm} (12)

Differently from the calculation of the thermal loads, the number of Gauss points is established in the entry data.

For the generic term of the \( K \) matrix, we have:
\[ K_{ij} = \int_{0}^{1} \int_{0}^{1} S \, K \, B_{ij} \, B_{ij} \, J \, d\xi \, d\eta \equiv \]
\[ \approx \sum_{k=1}^{N_GAU} S \, K \, B_{ij}(\xi_k, \eta_k) \, B_{ij}(\xi_k, \eta_k) \, J(\xi_k, \eta_k) \, PES_{(k)} \]  \hspace{1cm} (13)

with \( l = 1, 2 \).

Likewise, for the plane quadrangular element, one obtains:
\[ K_{ij} = \int_{-1}^{1} \int_{-1}^{1} S \, K \, B_{ij} \, B_{ij} \, J \, d\xi \, d\eta \equiv \]
\[ \approx \sum_{n=1}^{N_GAU} \sum_{m=1}^{N_GAU} S \, K \, B_{ij}(\xi_n, \eta_n) \, B_{ij}(\xi_n, \eta_n) \, J(\xi_n, \eta_n) \, PES_{(n)} \, PES_{(m)} \]  \hspace{1cm} (14)

The number of integration points depends also in this case on the type of problem and on the type of element.

Differently from the loads, the terms of the \( K \) matrix become polynomials only for triangular elements with straight walls and rectangular elements, since \( J \) is constant only in this way.

In these conditions, the degree of the polynomial to integrate is still a variable.

In the assialsymmetric case, the same rules may be used as in the plane case, when the element is small in comparison to the distance from the \( z \) axis.

If, instead, the elements are distorted, then the rules of integration will produce approximate values.

If the distortion is not excessive, the importance of the upper terms of \( J \) becomes scarcely influential.

One should note that the 4 x 4 rule of the quadrangular element with 8 assialsymmetric nodes has not been activated, assuming that the element is of a small dimension in comparison with the \( z \) distance ( \( r \) fairly constant).

By numerical integration, the upper triangle of the conductivity matrix is determined in the subroutines and then the lower triangle is constructed to have a perfectly symmetrical matrix.

2.1.4 Calculation of convection matrix

The calculation of the integral \( K^c_2 = \int_{\mathcal{S}^c} \left( \mathbf{H} \mathbf{N}^T \right)^T \, d\mathbf{S} \) is carried out in a special subroutine; in this case, the integral is mono-dimensional and therefore the integration involves the sole variable \( \xi \).

The generic term of the convection matrix in the plane case is:
\[ K_{ij} = \int_S s H N_i N_j dS \equiv \]
\[ \equiv \sum_{k=1}^{NGP} s H N_i (\xi_k) N_j (\xi_k) L(\xi_k) PES(k) \]

(15)

The integration points are selected on the basis of the usual scheme.

The remaining elements of the matrix relating to the internal nodes are, instead, void.

3. APPLICATIONS AND COMPARISONS

The obtained code will now be applied to the case described in [1] and subsequently in [3] and [4], in order to obtain an initial validation.

It should be emphasised that in [1], an isolated duct is considered, with aluminium plates \[ \lambda = 200 \text{ W} / (\text{m} \degree \text{C}) \], inserted in the duct, and heated by an electric resistance.

The experimental duct may have two wavy walls and is represented in Fig. 1:

![Fig. 1 - Two wavy walls](image)

or the duct has one wavy wall and is represented in Fig. 2

![Fig. 2 - One wavy wall](image)

In the third experiment, the duct is rectangular.

3.1 Exposure and control of the data in [1]

By using the above-mentioned code, it was possible to calculate the average convection coefficient for the situations described in [1].

For the purpose of continuing the comparison, the number of Nusselt (Nu) was graphed in Fig. 3a, b, c in function of the number of Reynolds (Re) and determined on the basis of the \[ h \] mean value calculated as an average of the local values for the three experiments, in function of increasing air flow rates (increasing Re) and for three different temperatures (60\degree, 80\degree, 100\degree) that are set on the central crest of the hot plate.
For precision, in the first graph the duct has two wavy walls (upper and lower) for a total area of the section equivalent to $260 \text{ cm}^2$.

The temperature of the air entering the duct was $20^\circ \text{C}$ and the calculation was made for flow rates of: $0.062 - 0.077 - 0.092 - 0.113 \text{ Kg/s}$.

In the second graph, the lower wall of the duct remains the same (wavy) as the previous calculation, while the upper one is plane, but in such a way as to maintain constant the total area of the cross section of the duct equal to $260 \text{ cm}^2$ (the equivalent diameter is, instead, different).

The temperature of the air in the experimentation was $18^\circ \text{C}$ and the values of the air flow rate remained the same as the previous calculation.

In the last case, a strip of pipe with a rectangular profile was considered, with the area of the cross section unchanged.

From the comparison, a perfect correspondence is deduced between experimental values and values resulting from the code, with an error less than approximately $2\%$.

One may deduce that the code would also provide reliable results in other cases, i.e. with various geometric situations and various fluids that flow into the duct at temperatures that are different from those experimented: the code represents a generalisation with respect to the experimental data.

![Graph of the number of Nusselt in function of the number of Reynolds in the first calculation](image)
3.2 Exposure and control of data in [3]

Now we will compare the code with the data reported in [3]. There are two series of experiments based on two different geometric configurations of the duct itself. For the comparison with the calculation code, we referred to the case of the two opposed sinusoidal profiles.

During the experiment, the distance between the principal values of the sinusoid of the two wavy walls was changed by 6 to 12 cm, with increases of 2 cm per time. Consequently, the air flow rate in the duct was varied to maintain constant the speed of the air; the power distributed to the electric resistances was adjustable in such a way as to maintain the temperature of the wavy wall of the aluminium plate to about 100 °C.

Since the temperature of the refrigerant fluid during the experiment was close to 22.5 °C, it is legitimate to point out that the heat transfer coefficient depended exclusively on the distance among the wavy walls.
The values of the convection coefficient, obtained in the experiment, are compared in Fig. 4 with the corresponding values calculated by the code and one may note the substantial agreement.

![Graph of the convection coefficient in function of the distance between the two wavy surfaces](image1)

**Fig. 4 - Graph of the convection coefficient in function of the distance between the two wavy surfaces**

### 3.3 Exposure and controls of data in [4]

Now let us compare the code with the data reported in [4].

The numerical simulations were conducted on two dissipators; the thermal exchange with the ambient occurs by natural convection, with the heat transfer coefficient dependent on the temperature of the fin and of the ambient. In particular, a length of 100 mm and a height of 60 mm is assumed for both dissipators; for the aluminium, $\lambda = 200 \text{ W/m °C}$ was set and the imposed thermal condition set at the support base of the dissipators was that of a constant thermal flow.

The comparison between the experimental and numerical data are reported in Fig. 5 and a substantial agreement may be noted.

![Graph of the thermal jump dissipator-ambient-1st and 2nd case](image2)

**Fig. 5 - Graph of the thermal jump dissipator-ambient-1st and 2nd case**
4. FURTHER VERIFICATION OF THE CODE IN THE CASE OF THE FIN

In the hypothesis of an isotropic and homogeneous solid in steady regime, the equation that describes
the heat transfer of a fin, as seen in [5], in the mono-dimensional case, is:

\[
\frac{d^2T}{dx^2} - \mu^2 (T - T_\infty) = 0
\]  

(16)

where \(\mu^2 = \frac{H}{\lambda A} \) and
\[A = \text{area of the cross section of the fin}\]
\[p = \text{perimeter of the fin}\]

(16) is a linear ordinary differential equation of the second order at constant coefficients, although it
becomes non-linear if \(\lambda\) depends on the temperature.

In the case in which the boundary conditions are \(T = T_0\) for \(x = 0\) and convection in \(x = L\), expressed by

\[-\lambda \frac{dT}{dx} = H(T - T_\infty),\]

the solution of (16) becomes:

\[
\frac{(T - T_\infty)}{(T_0 - T_\infty)} = \frac{\cosh \mu (L - x) + \left(\frac{\mu}{2}\right) \sinh \mu (L - x)}{\cosh \mu L + \left(\frac{\mu}{2}\right) \sinh \mu L}
\]

(17)

In the same way, in the assialsymmetric case, [5] is obtained for a fin having a constant thickness \(s\) and \(T_\infty = 0\) :

\[
\frac{d^2T}{dr^2} + \left(\frac{1}{r} \frac{dT}{dr}\right) - \mu^2 T = 0
\]

(18)

where \(\mu^2 = \frac{2H}{\lambda s}\)

Setting \(\mu r\) as an independent variable, (18) becomes a modified equation of Bessel of zero order,
whose general solution is given by:

\[
T = c_1 I_0(\mu r) + c_2 K_0(\mu r)
\]

(19)

where \(I_0\) and \(K_0\) are modified functions of Bessel of zero order, first and second kind.

Imposing the conditions to the boundary, if \(I_1\) and \(K_1\) are the functions of Bessel of the first order,
(19) becomes:

\[
T = T_0 \frac{I_0(\mu r)K_1(\mu b) + K_0(\mu r)I_1(\mu b)}{I_0(\mu a)K_1(\mu b) + K_0(\mu a)I_1(\mu b)}
\]

(20)

\(a = \text{internal radius; } b = \text{external radius.}\)

Then the calculation was analytically made and we proceeded by applying the code to assialsymmetric
fins having an internal radius \(a = 1\) cm, external \(b = 2\) cm, \(\mu^2 = 1\) and \(T_0 = 100^\circ\text{C.}\)

In this way it was verified that the patterns of the temperature calculated with the code were
practically convergent with the exact ones calculated with (20).

Fig. 6 shows the solution of the previous problem obtained with 11 nodes and compared with the
corresponding exact solution.
5. CONCLUSIONS

The code, although complex, did not reveal any particular problem in its use and permitted the determination of the value of the convection coefficient in various cases. From the comparisons carried out, the code proves to be in good agreement with the experimental data both as far as the calculated temperatures are concerned, and the courses of the same. In particular, it was verified that the average calculated convection coefficients, in fact, converge with those obtained experimentally.

Starting from the code, it is easy to obtain a generalisation of the desired values to use, for example, during the planning of apparatus. A possible extension of the code could also be that of adapting it to the calculation of the profile of temperatures in the case of the thermal transitories.

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Nomenclature

$B^e$: first derivatives matrix of shape functions
$F^e$: thermal loads vector
$F^e_0$: concentrated thermal loads vector
$F^e_H$: convection vector
$F^e_Q$: heat generation vector
$H$: heat transfer coefficient ($\text{W/m}^2\,\text{°C}$)
$K^e$: conductivity and convection matrix
$K^e_1$: conductivity matrix
$K^e_2$: convection matrix
$N^e$: shape functions matrix
$q_0$: thermal flow (W/m$^2$)
$Q$: heat generation (W/m$^3$)
$T$: temperature (°C)
$T_0$: imposed temperature (°C)
$T_\infty$: ambient temperature (°C)
\( \lambda \): thermal conductivity coefficient (W/m °C)

References


